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An exact completely positive programming formulation for the discrete ordered median problem: an extended version

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Abstract
This paper presents a first continuous, linear, conic formulation for the discrete ordered median problem (DOMP). Starting from a binary, quadratic formulation in the original space of location and allocation variables that are common in location analysis (L.A.), we prove that there exists a transformation of that formulation, using the same space of variables, that allows us to cast DOMP as a continuous, linear programming problem in the space of completely positive matrices. This is the first positive result that states equivalence between the family of continuous, convex problems and some hard combinatorial problems in L.A. The result is of theoretical interest because it allows us to share the tools from continuous optimization to shed new light into the difficult combinatorial structure of the class of ordered median problems that combines elements of the p-median, quadratic assignment and permutation polytopes.

Keywords Discrete ordered median problem · Completely positive reformulation · Conic linear programming

1 Introduction

The discrete ordered median problem (DOMP) represents a generalization of several well-known discrete location problems, such as p-median, p-center or $(k_1 + k_2)-$trimmed mean, among many others. DOMP provides a very flexible tool in accommodating actual aspects, as subsidized or compensation costs, in models of Logistics and Location, see Puerto and Fernández [22] and Nickel and Puerto [19]. This way different objective functions are cast within the same framework and similar tools can be used to solve problems with apparently different structure. The ordered median objective function computes ordered weighted averages of vectors. In the case when that objective function is applied to location problems those vectors correspond to distances or allocation costs from clients to service facilities. The problem was introduced in Nickel [18] and later studied by Kalcsics et al. [13], Nickel and Puerto [19], Boland et al. [3], Stanimirovic et al. [25], Marín et al. [17], Perea and Puerto

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[21] and Blanco et al. [2] among many other papers. DOMP is an NP-hard problem as an extension of the p-median problem.

Nickel [18] first presented a quadratic integer programming formulation for the DOMP. However, no further attempt to deal directly with that formulation was ever considered. Furthermore, that approach was never exploited in trying to find alternative reformulations or bounds; instead several linearizations in different spaces of variables (some of them rather promising) have been proposed to solve DOMP, Marín et al. [17] and Labbé et al. [16].

Motivated by the recent advances in conic optimization and the new tools that this branch of mathematical programming has provided for developing bounds and approximation algorithms for NP-hard problems (as for instance the max-cut problem and the quadratic assignment problem (QAP) among others [8]), we would like to revisit that earlier approach to DOMP. Our aim is to propose an alternative reformulation of the problem as a continuous, linear, conic optimization problem. Our interest is mainly theoretical and tries to borrow tools from continuous optimization to be applied in some discrete problems in the field of Location Analysis (L.A.). To the best of our knowledge reformulations of that kind have never been studied before for DOMP nor even in the wider field of L.A.

The goal of this paper is to prove that a natural binary, quadratically constrained, quadratic formulation for DOMP admits a reformulation as a continuous, linear optimization problem over the cone of completely positive matrices $\mathcal{C}^*$. Recall that a symmetric matrix $M \in \mathbb{R}^{n\times n}$ is called completely positive if it can be written as $M = \sum_{i=1}^{k} x_i x_i^t$ for some finite $k \in \mathbb{N}$ and $x_i \in \mathbb{R}^n_+$ for all $i = 1, \ldots, k$.

The paper is organized as follows. Section 2.1 formally defines the ordered median problem and sets its elements. Section 2.2 is devoted to describe a folk result that formulates the problem of sorting numbers as a feasibility problem in binary variables. This is used later as a building block to present the binary, quadratic, quadratically constrained formulation of DOMP. Section 3 contains the main result in this paper: DOMP is equivalent to a continuous, linear, conic optimization problem. Obviously, there are no shortcuts and the problem remains NP-hard but this new approach may contribute to increase the impact of conic optimization techniques in combinatorial optimization. The last section, namely Sect. 5 contains some conclusions and pointers to future research.

2 Definition and formulation of the problem

2.1 Problem definition

Let $S = \{1, \ldots, n\}$ denote the set of n sites. Let $C = (c_{j\ell})_{j,\ell=1,\ldots,n}$ be a given nonnegative $n \times n$ cost matrix, where $c_{j\ell}$ denotes the cost of satisfying demand point (client) j from a facility located at the site ℓ. We also assume the so called, *free self-service* situation, namely $c_{jj} = 0$ for all $j = 1, \ldots, n$. Let $p < n$ be the desired number of facilities to be located at the candidate sites. A solution to the facility location problem is given by a set $\mathcal{X} \subseteq S$ of p sites. The set of solutions to the location problem is therefore finite, although exponential in size, and it coincides with the $\binom{n}{p}$ subsets of size p of S.

We assume that each new facility has unlimited capacity. Therefore, each client j will be allocated to a facility located at the site ℓ of $\mathcal{X}$ with the lowest cost, i.e.

$$c_j = c_j(\mathcal{X}) := \min_{\ell \in \mathcal{X}} c_{j\ell}.$$

The costs for supplying clients, $c_1(\mathcal{X}), \ldots, c_n(\mathcal{X})$, are sorted in nondecreasing order. We define $\sigma_{\mathcal{X}}$ to be a permutation on $\{1, \ldots, n\}$ for which the inequalities

$$c_{\sigma_{\mathcal{X}}(1)}(\mathcal{X}) \leq \cdots \leq c_{\sigma_{\mathcal{X}}(n)}(\mathcal{X})$$

hold.

Now, for any nonnegative vector $\lambda \in \mathbb{R}^n_+$, DOMP consists of finding $\mathcal{X}^* \subset S$ with $|\mathcal{X}^*| = p$ such that:

$$\sum_{k=1}^{n} \lambda_k c_{\sigma_{\mathcal{X}^*}(k)}(\mathcal{X}^*) = \min_{\mathcal{X} \subset S, |\mathcal{X}|=p} \sum_{k=1}^{n} \lambda_k c_{\sigma_{\mathcal{X}}(k)}(\mathcal{X}).$$

2.2 Sorting as an integer program and a binary quadratic programming formulation of DOMP

For the sake of readability, we describe in the following a folk result that allows us to understand better the considered binary quadratic programming formulation for DOMP.

Let us assume that we are given n real numbers $r_1, \ldots, r_n$ and that we are interested in finding its sorting in nondecreasing sequence as a solution of a mathematical program. One natural way to do it is to identify the permutation that ensures such a sorting.

We can introduce the following *ordering* variables

$$P_{jk} = \begin{cases} 1 & \text{if the number } r_j \text{ is sorted in the } k\text{th position}, \quad j, k = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we can consider the following feasibility problem:

$$\min \quad 1 \tag{SORT}$$

$$\begin{aligned}
\text{s.t.} \ & \sum_{j=1}^{n} P_{jk} = 1, && \forall k = 1, \ldots, n, && (1)\\
& \sum_{k=1}^{n} P_{jk} = 1, && \forall j = 1, \ldots, n, && (2)\\
& \sum_{j=1}^{n} P_{jk} r_j \leq \sum_{j=1}^{n} P_{j,k+1} r_j, && \forall k = 1, \ldots, n-1, && \\
& P_{jk} \in \{0, 1\}, && \forall j, k = 1, \ldots, n. && (3)
\end{aligned}$$

Clearly, the constraints (1) and (2) model permutations and the constraint (3) ensures the appropriate order in any feasible solution since the number in sorted position k must be smaller than or equal to the one in position $k + 1$.

Now, we can insert in this formulation the elements of DOMP. Indeed, we have to replace the given real numbers $r_1, \ldots, r_n$ by the allocation costs induced by the location problem. Let X and $\mathcal{O}$ be the natural allocation and location variables in the location problem, namely

$$X_{j\ell} = \begin{cases} 1 & \text{if the demand point } j \text{ goes to the facility } \ell, \quad j, \ell = 1, \ldots, n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{O}_\ell = \begin{cases} 1 & \text{if a new facility is opened at site } \ell,\ \ell = 1,\ldots,n, \\ 0 & \text{otherwise.} \end{cases}$$

Based on the above variables, it follows that the cost induced by the allocation of the demand point j will be given by $\sum_{\ell=1}^{n} c_{j\ell} X_{j\ell}$. Thus, if we want a valid formulation of DOMP we can embed the location elements into the SORT formulation. This results in the following formulation (see [18]):

$$\min \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} P_{jk} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} \tag{MIQP-DOMP}$$

$$s.t. \sum_{j=1}^{n} P_{jk} = 1, \qquad \forall k = 1,\ldots,n, \tag{4}$$

$$\sum_{k=1}^{n} P_{jk} = 1, \qquad \forall j = 1,\ldots,n, \tag{5}$$

$$\sum_{j=1}^{n} \left(\sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} \right) P_{jk} \le \sum_{j=1}^{n} \left(\sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} \right) P_{jk+1}, \qquad \forall k = 1,\ldots,n-1, \tag{6}$$

$$\sum_{\ell=1}^{n} \mathcal{O}_\ell = p, \tag{7}$$

$$\sum_{\ell=1}^{n} X_{j\ell} = 1, \qquad \forall j = 1,\ldots,n, \tag{8}$$

$$\mathcal{O}_\ell \ge X_{j\ell}, \qquad \forall j, \ell = 1,\ldots,n, \tag{9}$$

$$P_{jk}, X_{j\ell}, \mathcal{O}_j \in \{0,1\}, \qquad \forall j, k, \ell = 1,\ldots,n. \tag{10}$$

Clearly, the objective function accounts for the ordered weighted sum of the allocation costs so that the cost sorted in position k is multiplied by λ_k for all $k = 1,\ldots,n$. The constraint (4) states that each position in the sequence from 1 to n will be occupied by one of the realized allocation costs; whereas the constraint (5) ensures that each allocation cost will be in one position between $1,\ldots,n$. The constraint (6) states that the cost in position k must be less than or equal to the one in position $k+1$. The next constraint, namely (7), sets that p facilities are open. The constraint (8) forces that each demand point is allocated to one facility and the constraint (9) ensures that the demand points can be assigned only to open facilities. Finally, (10) sets the range of the variables.

Based on our discussion, the above quadratically constrained, quadratic objective function formulation is valid for DOMP. We note in passing that one could relax the binary definition of the variables $\mathcal{O}_j \in \{0,1\}$ to simply nonnegativity, namely $\mathcal{O}_j \ge 0$, since boundedness of $\mathcal{O}$ is ensured by (7); and by (9) and the fact that $\mathcal{O}$ variables do not appear in the objective function, there will always exist binary optimal solutions. Therefore, in what follows (unless explicitly stated) and without loss of generality, $\mathcal{O}$ variables will be considered just nonnegative.

3 A completely positive reformulation of DOMP

The goal of this section is to developed a new reformulation for DOMP as a continuous, linear optimization problem in the cone of completely positive matrices $\mathcal{C}^*$. For this reason, we consider X, P and $\mathcal{O}$ as the original variables of the problem. These variables are standard in the already known formulations for DOMP.

For the ease of presentation, we use the following convention. We refer to the original variables in capital letters. The slack variables used to transform inequalities to equations in the formulations will be denoted by the Greek letters ζ, ξ, η. Finally, the elements of the matrix variables will be denoted by small letters. For instance, if $U = (u_{ij})$ is a matrix variable we refer to its elements as u_{ij}. In the following the *vec* operator stacks in a column vector the columns of a matrix and the *rvec* operator stacks in a column vector the rows of a matrix. Clearly, $rvec(A) = vec(A^T)$, i.e. *rvec* acting on a matrix A is equivalent to *vec* acting on the transpose of A. The $Diag$ operator maps a $n-$vector to a $n \times n$ diagonal matrix and the $diag$ operator extracts in a n-vector the diagonal of a $n \times n$-square matrix. We follow the convention that all single-index variables in our formulations are column vectors.

In the sequel rather than considering the standard version of DOMP in the literature of L.A., described above, we will consider an extended version of that problem because it permits to include more realistic aspects in the model. These modeling aspects come from the interaction between the different elements in the problems. That is, this extended model includes some extra interaction terms among the ordering variables ($P_{jk} P_{j'k'}$) and the allocation variables ($X_{j\ell} X_{pq}$), see e.g. Kalcsics et al. [14,15], Puerto et al. [23,24]. Let $D = (d_{jkj'k'}) \in \mathbb{R}^{n^2 \times n^2}$, where $d_{jkj'k'}$ is the interaction cost of sorting the demand point j in position k and the demand point j' in position k'. Moreover, let $H = (h_{j\ell pq}) \in \mathbb{R}^{n^2 \times n^2}$, where $h_{j\ell pq}$ is the cost incurred due to the allocation of the demand points j and p to the facilities ℓ and q, respectively, see e.g. Burkard [10]. The reader should observe that in the standard version of DOMP in the literature the cost matrices D and H satisfy $D = H = \Theta$, where Θ is the zero matrix.

In spite of the greater complexity coming from including new quadratic terms in the objective function of DOMP, our approach can handle this extended version of the problem with the same effort. Thus, in what follows we consider the following objective function for MIQP-DOMP.

$$\min \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} P_{jk} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} + 1/2 \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{j'=1}^{n} \sum_{k'=1}^{n} d_{jkj'k'} P_{jk} P_{j'k'}$$
$$+ 1/2 \sum_{j=1}^{n} \sum_{\ell=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} h_{j\ell pq} X_{j\ell} X_{pq} \tag{11}$$

Clearly, the objective function (11) accounts for the ordered weighted sum of the allocation costs so that the cost sorted in position k is multiplied by λ_k for all $k = 1, \ldots, n$, plus the interaction costs induced by the orderings and the allocations. This objective function replaces the usual one in MIQP-DOMP and allows us a more general analysis of the problem.

Our goal is to find new ways to represent the problem that may contribute to increase the impact of conic optimization techniques in combinatorial optimization. We show in the following that it is possible to cast DOMP within the field of continuous, linear, conic optimization and we derive an explicit reformulation.

Furthermore, we emphasize that this result is not straightforward. Contrary to the case of quadratic objective and linear constrained problems with some binary variables, where one can apply Burer's results [6, Theorem 2.6]; the above formulation is quadratic in the objective, in the constraints and also includes binary variables; and in our case the most recent, known sufficient conditions for obtaining conic reformulations do not directly apply [1,9,20]. In all cases, those results require some extra conditions on the considered quadratic constraints which in our case are not fulfilled: Burer and Dong [9] assume that the nonlinear constraints are in the form of the intersection of level sets of certain specific quadratic functions (see [9, Theorem 1]); Bai et al. [1] is applicable if the problem has just one nonconvex constraint being nonnegative over the domain defined by the remaining conic and linear constraints; and Peña et al. [20] require that the nonlinear constraints are nonnegative either on the level sets of some of the other nonlinear constraints [20, Theorem 4] or over the nonnegative orthant [20, Theorem 5]. None of those conditions can be ensured by the quadratic constraints (6) in our problem, namely $\sum_{j=1}^{n}\left(\sum_{\ell=1}^{n} c_{j\ell}X_{j\ell}\right)P_{jk} \leq \sum_{j=1}^{n}\left(\sum_{\ell=1}^{n} c_{j\ell}X_{j\ell}\right)P_{jk+1}, \forall k = 1,\ldots,n-1$. Moreover, although squaring them would result in nonnegative degree 4 polynomials which may allow to apply the results in Peña et al. [20], such a reformulation would require completely positive tensors (not matrices) implying a much larger set of variables than what it is used in our approach. In spite of that, we prove in our main result that such a reformulation does exist for DOMP. Thus, it links this problem to the general theory of convex, conic programming as a new instance of this broad class of problems. As a result, any new development arising in the field of convex, conic programming, could be readily transferred to DOMP advancing in its better understanding. The interested reader is referred to Dür [11], Bomze [4], Burer [7] and Xu and Burer [26] for a list of other examples of combinatorial problems that also fits to this framework.

In our approach, we first reformulate MIQP-DOMP as another quadratically constrained, quadratic programming problem that is instrumental in the construction of the final completely positive reformulation for DOMP. The latter reformulation is the main goal of this paper.

In our development we follow the approach by Burer [6] but we need to introduce modifications to go around the problematic quadratic constraint (6) that does not fit within that framework. In order to do that, we slightly modify the formulation MIQP-DOMP with the objective (11) and we introduce an additional set of variables W_k, $k = 1,\ldots,n$ which will represent the value of the cost that is sorted in position k. This means that $W_k = \sum_{j=1}^{n}\left(\sum_{\ell=1}^{n} c_{j\ell}X_{j\ell}\right)P_{jk}$ and although these variables are redundant they will simplify the presentation in our approach. Adding these equations for all $k = 1,\ldots,n$, results in another valid constraint, namely $\sum_{k=1}^{n} W_k = \sum_{j=1}^{n}\sum_{\ell=1}^{n} c_{j\ell}X_{j\ell}$, that we will also include in our formulation.

The following result provides a reformulation of MIQP-DOMP that is an important building block in the proof of the main result.

Lemma 3.1 *Problem MIQP1-DOMP is a reformulation of Problem MIQP-DOMP.*

$$\min \sum_{k=1}^{n} \lambda_k W_k + 1/2 \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{j'=1}^{n}\sum_{k'=1}^{n} d_{jkj'k'} P_{jk} P_{j'k'} \qquad \text{(MIQP1-DOMP)}$$
$$+ 1/2 \sum_{j=1}^{n}\sum_{\ell=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} h_{j\ell pq} X_{j\ell} X_{pq}$$

$$s.t. (4), (5), (7), (8),$$
$$X_{j\ell} - \mathcal{O}_\ell + \zeta_{j\ell} = 0, \quad \forall j, \ell = 1,\ldots,n, \qquad (12)$$

$$W_k - \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} + \sum_{\ell=1}^{n} c_{j\ell}(1 - P_{jk}) - \eta_{jk} = 0, \quad \forall j, k = 1, \ldots, n, \tag{13}$$

$$W_k = \sum_{j=1}^{n} P_{jk} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell}, \quad \forall k = 1, \ldots, n, \tag{14}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} (P_{jk} - P_{jk}^2) = 0, \tag{15}$$

$$\sum_{j=1}^{n} \sum_{\ell=1}^{n} (X_{j\ell} - X_{j\ell}^2) = 0, \tag{16}$$

$$W_k - W_{k+1} + \xi_k = 0, \quad \forall k = 1, \ldots, n-1, \tag{17}$$

$$\sum_{k=1}^{n} W_k - \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} = 0, \tag{18}$$

$$P_{jk}, X_{j\ell} \in \{0, 1\}, \ \mathcal{O}_\ell, W_k, \xi_j, \eta_{jk}, \zeta_{j\ell} \geq 0, \quad \forall j, k, \ell = 1, \ldots, n. \tag{19}$$

Proof In order to get the formulation in the statement of the lemma, we begin by considering an intermediate reformulation of MIQP-DOMP which results by augmenting the variables $W_k = \sum_{j=1}^{n} P_{jk} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell}$, for all $k = 1, \ldots, n$ (14). In addition, we add three redundant constraints. The first two are quadratic, namely $\sum_{j=1}^{n} \sum_{k=1}^{n} (P_{jk} - P_{jk}^2) = 0$ (15) and $\sum_{j=1}^{n} \sum_{\ell=1}^{n} (X_{j\ell} - X_{j\ell}^2) = 0$ (16); and the third one is linear $\sum_{k=1}^{n} W_k - \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} = 0$ (18). These three constraints will be instrumental in our developments. It is clear that augmenting the above variables and constraints to MIQP-DOMP results in a reformulation (see MIQP2-DOMP below) since from every feasible solution to one problem one can obtain a feasible solution to the other problem with the same objective value.

We write the new problem explicitly for the sake of readability. Starting from MIQP–DOMP, with the new objective function (11), we perform the transformation described above and the new formulation is:

$$\begin{aligned} \min \sum_{k=1}^{n} \lambda_k W_k &+ 1/2 \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{j'=1}^{n} \sum_{k'=1}^{n} d_{jkj'k'} P_{jk} P_{j'k'} \\ &+ 1/2 \sum_{j=1}^{n} \sum_{\ell=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} h_{j\ell pq} X_{j\ell} X_{pq} \end{aligned} \tag{MIQP2-DOMP}$$

$$\begin{aligned} s.t.&(4), (5), (6), (7), (8), (12), (14), (15), (16), (18), \\ &P_{jk}, X_{j\ell} \in \{0, 1\}, \ \mathcal{O}_\ell, W_k, \zeta_{j\ell} \geq 0, \qquad \forall j, k, \ell = 1, \ldots, n. \end{aligned} \tag{20}$$

MIQP2-DOMP is a mixed-{0, 1}, quadratically constrained, quadratic objective problem. From this formulation, we can obtain the one in the statement of the lemma by: 1) reformulating constraints (6) as $W_k - W_{k+1} - \xi_k = 0$ for all $k = 1, \ldots, n-1$, (17); 2) adding to the formulation the valid inequalities (13), i.e $W_k - \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} + \sum_{\ell=1}^{n} c_{j\ell}(1 - P_{jk}) - \eta_{jk} = 0, \quad \forall j, k = 1, \ldots, n.$

We observe that the constraint (13) is valid by the following argument. First of all, if $P_{jk} = 0$ then (13) reduces to $W_k \geq \sum_{\ell=1}^{n} c_{j\ell}(X_{j\ell} - 1)$ which always holds since $W_k \geq 0$ and $X_{j\ell} \leq 1$ for all $\ell = 1, \ldots, n$. Second, if $P_{jk} = 1$ then by the definition of W_k, $W_k = \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} \geq \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell}$.

Augmenting these constraints to MIQP2-DOMP results exactly in MIQP1-DOMP and the result follows.

□

Additionally and for the sake of readability we will introduce the following notation. We set the entire family of variables of the problem MIQP1-DOMP as the column vector ϕ given by

$$\phi^T = (rvec(P)^T, rvec(X)^T, \mathcal{O}^T, W^T, \xi^T, rvec(\eta)^T, rvec(\zeta)^T). \tag{21}$$

Recall that we assume that all single-index variables, namely $\mathcal{O}$, W, ξ, are column vectors. Furthermore, to alleviate the notation, we will denote the tuple of vector of coefficients of a constraint numbered as (#), in the formulation MIQP1-DOMP, by $[a_{(\#)}]$. For instance, if the reference (#) defines a set of constraints for all $\ell = 1, \ldots, n$ then we refer to the vector of coefficients in the ℓth constraint in that set by $[a_{(\#)}]_\ell$. Analogously, with $b_{(\#)\ell}$ we refer to its right-hand-side. For instance, $[a_{(4)}]_1^T \phi = b_{(4)1}$ refers to the first equation of constraints (4) which is $\sum_{j=1}^{n} P_{j1} = 1$.

Next, we consider the matrix form $\Phi = \phi\phi^T$ that is symmetric with the upper triangular part given by:

$$\Phi = \begin{array}{c} \begin{array}{ccccccc} n^2 & n^2 & n & n & n & n^2 & n^2 \end{array} \\ \begin{bmatrix} Q & V & P\mathcal{O} & PW & P\xi & P\eta & P\zeta \\ & U & X\mathcal{O} & XW & X\xi & X\eta & X\zeta \\ & & \Sigma & \mathcal{O}W & \mathcal{O}\xi & \mathcal{O}\eta & \mathcal{O}\zeta \\ & & & \Omega & W\xi & W\eta & W\zeta \\ & & & & \Psi & \xi\eta & \xi\zeta \\ & & & & & \Pi & \eta\zeta \\ & & & & & & \mathcal{Z} \end{bmatrix} \end{array} \in \mathbb{R}^{(4n^2+3n)\times(4n^2+3n)} \tag{22}$$

Each entry in Φ is a block of variables. The different submatrices that appear in Φ are described below:

$$Q = rvec(P)rvec(P)^T = \begin{bmatrix} rvec(P)P_{11} \\ \vdots \\ rvec(P)P_{nn} \end{bmatrix} = (q_{ikj\ell}) \in \mathbb{R}^{n^2 \times n^2}.$$

There exist structural conditions on the product of variables $P_{ik}P_{j\ell}$ since $diag(Q) = rvec(P)$.

The matrix U is analogous replacing P by X. Thus,

$$U = rvec(X)rvec(X)^T = \begin{bmatrix} rvec(X)X_{11} \\ \vdots \\ rvec(X)X_{nn} \end{bmatrix} = (u_{ikj\ell}) \in \mathbb{R}^{n^2 \times n^2}.$$

There are also conditions on the product of variables $X_{ik}X_{j\ell}$ since $diag(U) = rvec(X)$.

Next, the matrix V is

$$V = rvec(P)rvec(X)^T = \begin{bmatrix} rvec(X)P_{11} \\ \vdots \\ rvec(X)P_{nn} \end{bmatrix} = (v_{ikj\ell}) \in \mathbb{R}^{n^2 \times n^2}.$$

The remaining submatrices that are required because their terms appear in some constraints are defined accordingly:

$\Sigma = \mathcal{O}\mathcal{O}^T = (\sigma_{ij}) \in \mathbb{R}^{n\times n}$,
$\Omega = WW^T = (\omega_{ij}) \in \mathbb{R}^{n\times n}$, $\Psi = \xi\xi^T = (\psi_{ij}) \in \mathbb{R}^{n\times n}$,
$\Pi = rvec(\eta)rvec(\eta)^T = (\pi_{ikj\ell}) \in \mathbb{R}^{n^2\times n^2}$, $\mathcal{Z} = rvec(\zeta)rvect(\zeta)^T = (z_{ijrs}) \in \mathbb{R}^{n^2\times n^2}$,
$(P\mathcal{O}) = rvec(P)\mathcal{O}^T = ((P\mathcal{O})_{ij\ell}) \in \mathbb{R}^{n^2\times n}$, $(PW) = rvec(P)W^T = (\rho_{ikj}) \in \mathbb{R}^{n^2\times n}$,
$(P\xi) = rvec(P)\xi^T = ((P\xi)_{ij\ell}) \in \mathbb{R}^{n^2\times n}$, $(P\eta) = rvec(P)rvec(\eta)^T = (\upsilon_{ijrs}) \in \mathbb{R}^{n^2\times n^2}$,
$(P\zeta) = rvec(P)rvect(\zeta)^T = ((P\zeta)_{ijrs}) \in \mathbb{R}^{n^2\times n^2}$,
$(X\mathcal{O}) = rvec(X)\mathcal{O}^T = (\chi_{ikj}) \in \mathbb{R}^{n^2\times n}$, $(XW) = rvec(X)W^T = (\gamma_{ik\ell}) \in \mathbb{R}^{n^2\times n}$,
$(X\eta) = rvec(X)rvec(\eta)^T = (\kappa_{ijrs}) \in \mathbb{R}^{n^2\times n^2}$, $(X\zeta) = rvec(X)rvect(\zeta)^T = (\tau_{ijrs}) \in \mathbb{R}^{n^2\times n^2}$,
$(\mathcal{O}\zeta) = \mathcal{O}rvect(\zeta)^T = (\beta_{\ell rs}) \in \mathbb{R}^{n\times n^2}$,
$(W\xi) = W\xi^T = (\delta_{ij}) \in \mathbb{R}^{n\times n}$, $(W\eta) = Wrvec(\eta)^T = (\epsilon_{irs}) \in \mathbb{R}^{n\times n^2}$.

Finally, consider the coefficient matrix G and the vector g indexed in the same basis as ϕ.

$$G = \begin{bmatrix} D & F & \Theta & \Theta & \Theta & \Theta & \Theta \\ F^T & H & \Theta & \Theta & \Theta & \Theta & \Theta \\ \Theta & \Theta & \Theta & \Theta & \Theta & \Theta & \Theta \\ \Theta & \Theta & \Theta & \Theta & \Theta & \Theta & \Theta \\ \Theta & \Theta & \Theta & \Theta & \Theta & \Theta & \Theta \\ \Theta & \Theta & \Theta & \Theta & \Theta & \Theta & \Theta \\ \Theta & \Theta & \Theta & \Theta & \Theta & \Theta & \Theta \end{bmatrix} \text{ where } F = \begin{bmatrix} \overbrace{\lambda_1 \otimes C_{1\cdot}}^{X_{1\cdot}} & \overbrace{\Theta}^{X_{2\cdot}} & \dots & \overbrace{\Theta}^{X_{n\cdot}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n \otimes C_{1\cdot} & \Theta & \dots & \Theta \\ \Theta & \lambda_1 \otimes C_{2\cdot} & \dots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \lambda_n \otimes C_{2\cdot} & \dots & \Theta \\ \Theta & \Theta & \dots & \lambda_1 \otimes C_{n\cdot} \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \dots & \lambda_n \otimes C_{n\cdot} \end{bmatrix}$$

and

$$g^T = [\overbrace{\Theta}^{P}, \overbrace{\Theta}^{X}, \overbrace{\Theta}^{\mathcal{O}}, \overbrace{\lambda^T}^{W}, \overbrace{\Theta}^{\xi}, \overbrace{\Theta}^{\eta}, \overbrace{\Theta}^{\zeta}].$$

Recall that, as already introduced in Sect. 3, Θ stands for the matrix of the adequate size with all its entries equal to zero.

Clearly,

$$1/2\langle G, \Phi\rangle = 1/2trace(G\Phi) = \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} P_{jk} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} + 1/2 \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{j'=1}^{n}\sum_{k'=1}^{n} d_{jkj'k'} P_{jk} P_{j'k'}$$
$$+ 1/2 \sum_{j=1}^{n}\sum_{\ell=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} h_{j\ell pq} X_{j\ell} X_{pq}, \tag{23}$$

is the objective function of Problem MIQP1-DOMP. Analogously,

$$\langle F, V\rangle = trace(FV) = \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} P_{jk} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} = \lambda^T W = \ g^T \phi. \tag{24}$$

One can check that the above expression is the first term in the objective function of Problem MIQP1-DOMP.

Let $\mathcal{L}$ be the set defined by the linear constraints of MIQP1-DOMP, namely

$$\mathcal{L} = \{\phi \geq 0 : \phi \text{ satisfies } (4), (5), (7), (8), (12), (13), (17) \text{ and } (18)\}, \tag{25}$$

and let $A\phi = b$ be system of equations that describes the set $\mathcal{L}$. Now, we consider a new transformation of MIQP1-DOMP using the matrix variable Φ. This transformation requires: 1) the linear constraints, $\mathcal{L}$, that come from MIQP1-DOMP, namely $A\phi = b$, 2) the squares of those constraints in the matrix variable Φ, namely $diag(A\Phi A^T) = b \circ b$, where $\circ$ is the Hadamard product of vectors, 3) the quadratic constraints of MIQP1-DOMP written in the matrix variable Φ:

$$W_k - \sum_{j,\ell=1}^{n} c_{j\ell} v_{jkj\ell} = 0, \ \forall k = 1, \ldots, n, \tag{26}$$

$$\sum_{j,k=1}^{n} (P_{jk} - q_{jkjk}) = 0, \tag{27}$$

$$\sum_{j,\ell=1}^{n} (X_{j\ell} - u_{j\ell j\ell}) = 0, \tag{28}$$

and 4) the matrix

$$\overline{\Phi} = \begin{pmatrix} 1 & \phi^T \\ \phi & \Phi \end{pmatrix} \in \mathcal{C}^*,$$

where $\mathcal{C}^*$ is the cone of completely positive matrices of the appropriate dimension (see [6, Theorem 2.6]). Schematically, we can write that formulation as follows:

$$\begin{aligned}
\min \ & \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{j'=1}^{n}\sum_{k'=1}^{n} d_{jkj'k'} q_{jkj'k'} + 1/2 \sum_{j=1}^{n}\sum_{\ell=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} h_{j\ell pq} u_{j\ell pq} \qquad \text{(CP-DOMP-0)}\\
& + 1/2 \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n}\sum_{\ell=1}^{n} c_{j\ell} v_{jkj\ell}\\
\text{s.t. } & \phi \in \mathcal{L},\\
& diag(A^T \Phi A) = b \circ b, \ \text{where } A \text{ is the matrix of } \mathcal{L},\\
& W_k - \sum_{j,\ell=1}^{n} c_{j\ell} v_{jkj\ell} = 0, \ \forall k = 1, \ldots, n,\\
& \sum_{j,k=1}^{n} (P_{jk} - q_{jkjk}) = 0,\\
& \sum_{j,\ell=1}^{n} (X_{j\ell} - u_{j\ell j\ell}) = 0,
\end{aligned}$$

$$\overline{\Phi} = \begin{pmatrix} 1 & \phi^T \\ \phi & \Phi \end{pmatrix} \in \mathcal{C}^*.$$

It is well-known that CP-DOMP-0 is always a relaxation of MIQP1-DOMP and therefore it is also a relaxation for DOMP. Our next result proves that actually it is not a relaxation but a reformulation.

Theorem 3.1 *DOMP belongs to the class of continuous, convex, conic optimization problems. Problem CP-DOMP-0 is equivalent to MIQP1-DOMP, i.e.: (i) they have equal optimal objective value, (ii) if (ϕ^*, Φ^*) is a feasible solution for Problem CP-DOMP-0 then ϕ^* is in the convex hull of feasible solutions of Problem MIQP1-DOMP.*

Proof Clearly, the objective function of the problem CP-DOMP-0 can be written as a linear form of Φ (see (23)):

$$\langle F, V\rangle + 1/2\langle D, Q\rangle + 1/2\langle H, U\rangle = 1/2\langle G, \Phi\rangle.$$

Define $\mathcal{L}$ as in (25), and let $\mathcal{L}_\infty$ be the recession cone of $\mathcal{L}$.

We claim that $\mathcal{L}_\infty = 0$ since $\mathcal{L}$ is bounded. Indeed, we observe that P and X are non-negative by definition and bounded above by 1 since they satisfy (4) and (8), respectively. The variables $\mathcal{O}$ are also bounded from below by construction and bounded from above by (7). Finally, for all $k = 1, \ldots, n$ the variables W_k are also bounded above since they are nonnegative and $W_k \le n \max_{j,\ell} c_{j\ell}$ by (8) and (18). This proves that all variables $P, X, \mathcal{O}$ and W are nonnegative and bounded above. To prove that also the slack variables η, ξ and ζ are bounded we proceed as follows. We observe from (13) that $\eta_{jk} \le W_k + \sum_{\ell=1}^{n} c_{j\ell}$, for all $j, k = 1, \ldots, n$ and thus variables η are bounded. Analogously, from (17), we get that $\xi_k \le W_{k+1}$, for all $k = 1, \ldots, n-1$, hence ξ variables are bounded as well. Finally, using (12), it follows that $\zeta_{j\ell} \le \mathcal{O}_\ell$, for all $j, \ell = 1, \ldots, n$, and thus using the boundedness of $\mathcal{O}$ the slack variables ζ are also bounded.

Therefore, it follows that the recession cone of the linear part of the feasible region of MIQP1-DOMP is the zero vector.

Next, we consider the sets:

$$\begin{aligned}
\mathcal{L}' =& \mathcal{L} \cap \{\phi : W_k - \sum_{j,\ell=1}^{n} c_{j\ell} P_{jk} X_{j\ell} = 0,\ k = 1, \ldots, n,\ \sum_{j,k=1}^{n} (P_{jk} - P_{jk}^2) = 0, \\
& \sum_{j,\ell=1}^{n} (X_{j\ell} - X_{j\ell}^2) = 0\}, \\
(\mathcal{L}')^1 =& \left\{ \begin{pmatrix} 1 \\ \phi \end{pmatrix} \begin{pmatrix} 1 \\ \phi \end{pmatrix}^T : \phi \in \mathcal{L}' \right\}, \\
\mathcal{R} =& \left\{ \overline{\Phi} = \begin{pmatrix} 1 & \phi^T \\ \phi & \Phi \end{pmatrix} \in \mathcal{C}^* : A\phi = b,\ \ diag(A\Phi A^T) = b \circ b \right\}, \\
\mathcal{R}' =& \mathcal{R} \cap \Bigg\{ \overline{\Phi} = \begin{pmatrix} 1 & \phi^T \\ \phi & \Phi \end{pmatrix} : W_k - \sum_{j,\ell=1}^{n} c_{j\ell} v_{jkj\ell} = 0, \forall k = 1, \ldots, n,\ \sum_{j,k=1}^{n} (P_{jk} - q_{jkjk}) = 0, \\
& \sum_{j,\ell=1}^{n} (X_{j\ell} - u_{j\ell j\ell}) = 0 \Bigg\}.
\end{aligned}$$

Recall the $A\phi = b$ is the set of linear constraints that describe the set $\mathcal{L}$, $\mathcal{L}'$ is the feasible region of MIQP1-DOMP and $\mathcal{R}'$ is the feasible region of CP-DOMP-0.

We prove, that

$$conv((\mathcal{L}')^1) = \mathcal{R}'.$$

The inclusion $conv((\mathcal{L}')^1) \subseteq \mathcal{R}'$ is clear. For the reverse inclusion, since $\mathcal{R}' \subset \mathcal{R} \subset \mathcal{C}^*$ then for any matrix $\overline{\Phi} \in \mathcal{R}'$, it is known (see e.g. Burer [6]) that there exists a representation as:

$$\overline{\Phi} = \begin{pmatrix} 1 & \phi^T \\ \phi & \Phi \end{pmatrix} = \sum_{r\in I} \mu^r \begin{pmatrix} 1 \\ \phi^r \end{pmatrix}\begin{pmatrix} 1 \\ \phi^r \end{pmatrix}^T + \sum_{r\in J} \begin{pmatrix} 0 \\ \gamma^r \end{pmatrix}\begin{pmatrix} 0 \\ \gamma^r \end{pmatrix}^T,$$

where I and J are finite sets of indices, $\mu^r \geq 0$ for all $r \in I$, $\sum_{r\in I} \mu^r = 1$, $\phi^r \in \mathcal{L}$ for all $r \in I$ and $\gamma^r \in \mathcal{L}_\infty$ for all $r \in J$. Therefore, since the recession cone $\mathcal{L}_\infty$ is the zero vector, the representation above reduces to:

$$\overline{\Phi} = \begin{pmatrix} 1 & \phi^T \\ \phi & \Phi \end{pmatrix} = \sum_{r\in I} \mu^r \begin{pmatrix} 1 \\ \phi^r \end{pmatrix}\begin{pmatrix} 1 \\ \phi^r \end{pmatrix}^T, \tag{29}$$

with $\mu^r \geq 0$ for all $r \in I$, $\sum_{r\in I} \mu^r = 1$, and $\phi^r \in \mathcal{L}$ for all $r \in I$.

We claim that $\sum_{j,k=1}^n (P_{jk} - q_{jkjk}) = 0$, $\sum_{j,\ell=1}^n (X_{j\ell} - u_{j\ell j\ell}) = 0$ and $W_k - \sum_{j,\ell=1}^n c_{j\ell} v_{jkj\ell} = 0, \forall k = 1, \dots, n$ implies that each ϕ^r with $r \in I$ satisfies $\sum_{j,k=1}^n (P_{jk}^r - (P_{jk}^r)^2) = 0$, $\sum_{j,\ell=1}^n (X_{j\ell}^r - (X_{j\ell}^r)^2) = 0$ and $W_k^r - \sum_{j,\ell=1}^n c_{j\ell} P_{jk}^r X_{j\ell}^r = 0, \;\forall\, k = 1, \dots, n$. This will complete the proof.

First of all, we observe that, by the representation (29), it holds that: $P_{jk} = \sum_{r\in I} \mu^r P_{jk}^r$, $X_{j\ell} = \sum_{r\in I} \mu^r X_{j\ell}^r$, $W_k = \sum_{r\in I} \mu^r W_k^r$, $q_{jkjk} = \sum_{r\in I} \mu^r (P_{jk}^r)^2$ and $v_{jkj\ell} = \sum_{r\in I} \mu^r P_{jk}^r X_{j\ell}^r$; for all $j, k, \ell = 1, \dots, n$.

We begin with $\sum_{j,k=1}^n (P_{jk} - q_{jkjk}) = 0$ and replace the variables P_{jk} and q_{jkjk} by their representation in terms of (29) which results in:

$$0 = \sum_{j=1}^n \sum_{k=1}^n \left(\sum_{r\in I} \mu^r P_{jk}^r - \sum_{r\in I} \mu^r (P_{jk}^r)^2 \right) = \sum_{j=1}^n \sum_{k=1}^n \left(\sum_{r\in I} \mu^r (P_{jk}^r - (P_{jk}^r)^2) \right). \tag{30}$$

It is clear that $P_{jk}^r \leq 1$ and $X_{j\ell}^r \leq 1$ for all $j, k, \ell = 1, \dots, n$. Indeed, $P_{jk} = \sum_{r\in I} \mu^r P_{jk}^r$ with $P_{jk}^r \geq 0$ for all $r \in I$ and $P_{jk} \leq 1$ since it satisfies (4) and (5). Therefore $P_{jk}^r \leq 1$, for all $j, k = 1, \dots, n$, then all the addends in (30) are nonnegative and this implies that

$$P_{jk}^r - (P_{jk}^r)^2 = 0, \qquad \forall\, j, k = 1, \dots, n,\; r \in I. \tag{31}$$

The proof for $\sum_{j,\ell=1}^n (X_{j\ell}^r - (X_{j\ell}^r)^2) = 0$ is similar, but to prove that $X_{j\ell}^r \leq 1$ we use the inequality (8) valid for $X_{j\ell}$ instead of (4) and (5). This proves the first part of the claim.

Next, we consider $W_k - \sum_{j,\ell=1}^n c_{j\ell} v_{jkj\ell} = 0$. Then, we replace W_k and $v_{jkj\ell}$ by their representation in terms of (29) which results in

$$0 = \sum_{r\in I} \mu^r W_k^r - \sum_{j=1}^n \sum_{\ell=1}^n \sum_{r\in I} \mu^r c_{j\ell} P_{jk}^r X_{j\ell}^r = \sum_{r\in I} \mu^r \left(W_k^r - \sum_{j=1}^n \sum_{\ell=1}^n c_{j\ell} P_{jk}^r X_{j\ell}^r \right). \tag{32}$$

Now, since W_k^r, $X_{j\ell}^r$ and P_{jk}^r satisfy (13) for all $r \in I$ and $P_{jk}^r \geq 0$ for all j, k and r, we multiply both sides of the inequality (13) by P_{jk}^r. This results in:

$$W_k^r P_{jk}^r \geq \sum_{\ell=1}^n c_{j\ell} X_{j\ell}^r P_{jk}^r - \sum_{\ell=1}^n c_{j\ell} P_{jk}^r (1 - P_{jk}^r), \quad \forall j, k = 1, \dots, n,\; r \in I.$$

We sum the above inequalities for all $j = 1, \ldots, n$ to obtain:

$$W_k^r \overbrace{\sum_{j=1}^n P_{jk}^r}^{=1\,(4)} \geq \sum_{j=1}^n \sum_{\ell=1}^n c_{j\ell} X_{j\ell}^r P_{jk}^r - \sum_{j=1}^n \sum_{\ell=1}^n c_{j\ell} \overbrace{(P_{jk}^r - (P_{jk}^r)^2)}^{=0\,(31)}, \quad \forall k = 1, \ldots, n, \ r \in I.$$

Hence, we obtain:

$$W_k^r \geq \sum_{j=1}^n \sum_{\ell=1}^n c_{j\ell} X_{j\ell}^r P_{jk}^r, \quad \forall k = 1, \ldots, n, \ r \in I. \tag{33}$$

This condition combined with (32) proves the second part of the claim. Therefore, $\mathcal{R}' \subseteq conv((\mathcal{L}')^1)$ also holds, and thus, $\mathcal{R}' = conv((\mathcal{L}')^1)$.

The above property allows us to rewrite Problem MIQP1-DOMP as an equivalent continuous, convex, conic linear optimization problem in the matrix variables $\begin{pmatrix} 1 & \phi^T \\ \phi & \Phi \end{pmatrix}$ as described in CP-DOMP-0. □

The feasible region of the formulation CP-DOMP-0 always has an empty interior [6]. However, we can remove the explicit dependency on the original variables ϕ obtaining a new formulation that only depends on the essential matrix variable Φ and that, according to Burer [6], may enjoy having a nonempty interior.

Theorem 3.2 *Problem CP-DOMP-0 can be reformulated removing the explicit dependency on the ϕ variables as:*

$$\min \sum_{k=1}^n \lambda_k \sum_{j=1}^n \sum_{\ell=1}^n c_{j\ell} v_{jkj\ell} + 1/2 \sum_{j=1}^n \sum_{k=1}^n \sum_{j'=1}^n \sum_{k'=1}^n d_{jkj'k'} q_{jkj'k'} \tag{CP-DOMP}$$

$$+ 1/2 \sum_{j=1}^n \sum_{\ell=1}^n \sum_{p=1}^n \sum_{q=1}^n h_{j\ell pq} u_{j\ell pq}$$

$$s.t.\ \Phi[a_{(4)}]_1 \in \mathcal{L},$$

$$diag(A^T \Phi A) = b \circ b, \ \text{where } A \text{ is the matrix of } \mathcal{L},$$

$$\sum_{\ell=1}^n \rho_{1\ell k} - \sum_{j=1}^n \sum_{\ell=1}^n c_{j\ell} v_{jkj\ell} = 0, \ \forall k = 1, \ldots, n,$$

$$\sum_{j,k=1}^n \left(\sum_{\ell=1}^n q_{jk1\ell} - q_{jkjk} \right) = 0,$$

$$\sum_{j,\ell=1}^n \left(\sum_{r=1}^n v_{j\ell 1r} - u_{j\ell j\ell} \right) = 0,$$

$$\Phi \in \mathcal{C}^*.$$

Moreover, if $\hat{\Phi}$ is a feasible solution for Problem CP-DOMP then $\hat{\Phi}[a_{(4)}]_1$ is in the convex hull of feasible solutions of Problem MIQP1-DOMP.

Proof The proof is based on finding an appropriate linear combination of the rows of the linear constraints in $\mathcal{L}$, namely $A\phi = b$, that allows one a writing of ϕ as a linear combination of Φ, following an argument similar to the one in Burer [6]. We give full details of this argument for the sake of completeness.

Let IC be the index set of the different families of constraints of MIQP1-DOMP and for each $i \in IC$, let JC_i be the index set of the constraints in the ith family.

Take the first equation in (4), i.e. $\sum_{j=1}^{n} P_{j1} = 1$. Recall that we rewrite this constraint as $[a_{(4)}]_1^T \phi = b_{(4)1}$. Next, define the vector $\beta = (\beta_{ij})_{\left(\substack{i\in IC\\ j\in JC_i}\right)}$, as $\beta_{11} = 1$, $\beta_{ij} = 0$, for all $(i, j) \neq (1, 1)$.

It is clear that $\sum_{i\in IC} \sum_{j\in JC_i} \beta_{ij}[a_{(i)}]_j = [a_{(4)}]_1 \geq 0$ and that $\sum_{i\in IC} \sum_{j\in JC_i} \beta_{ij} b_{(i)j} = 1$. Let us denote $\alpha = \sum_{i\in IC} \sum_{j\in JC_i} \beta_{ij}[a_{(i)}]_j$. Then,

$$\alpha^T \phi = \sum_{i\in IC} \sum_{j\in JC_i} \beta_{ij}[a_{(i)}]_j^T \phi = \sum_{i\in IC} \sum_{j\in JC_i} \beta_{ij} b_{(i)j} = 1.$$

Thus, since $\phi\phi^T = \Phi$, from the above we obtain

$$\phi\phi^T \alpha = \Phi\alpha = \phi. \tag{34}$$

On the other hand, $1 = \alpha^T \phi\phi^T \alpha = \alpha^T \Phi\alpha$, and hence

$$\overline{\Phi} := \begin{bmatrix} 1 & \phi^T \\ \phi & \Phi \end{bmatrix} = \begin{bmatrix} 1 & \alpha^T \Phi \\ \Phi\alpha & \Phi \end{bmatrix} = (\alpha\ I)^T \Phi(\alpha I).$$

Thus, since $\alpha \geq 0$, if $\Phi \in \mathcal{C}^*$ then $\overline{\Phi} \in \mathcal{C}^*$. To prove the converse, it suffices to recall that the principal submatrices of completely positive matrices are completely positive. Therefore, if $\overline{\Phi} \in \mathcal{C}^*$ then $\Phi \in \mathcal{C}^*$.

To obtain the formulation (CP-DOMP) we only need to replace any occurrence of ϕ in CP-DOMP-0 by $\Phi[a_{(4)}]_1$ and observe that, in terms of $\Phi[a_{(4)}]_1$, $W_k = \sum_{\ell=1}^{n} \rho_{1\ell k}$, for all $k = 1, \dots, n$, $P_{jk} = \sum_{\ell=1}^{n} q_{jk1\ell}$, for all $j, k = 1, \dots, n$ and $X_{j\ell} = \sum_{r=1}^{n} v_{j\ell 1r}$, for all $j, \ell = 1, \dots, n$.

To complete the proof we apply the assertion (ii) of Theorem 3.1.

Summarizing, we have obtained that the formulation CP-DOMP is exact for DOMP and is linear in $\Phi \in \mathcal{C}^*$. □

An explicit reformulation of CP-DOMP, namely (CP-DOMP-Explicit), can be found in the "Appendix".

4 Further observations

This section gathers some remarks that show additional properties of the problem MIQP–DOMP or its reformulations. We start by showing a set of inequalities that can be used to replace (9), provided that $\mathcal{O}_\ell$ are considered binary variables.

Proposition 4.1 *Let us assume that the set of inequalities* (4)–(10) *hold and the variables* $\mathcal{O}_\ell$ *are binary. Then, the set of inequalities* (9) *can be replaced by*

$$\sum_{j=1}^{n} X_{j\ell} - (n - p + 1)\mathcal{O}_\ell \leq 0, \quad \forall \ell = 1, \dots, n \tag{35}$$

without affecting the optimal solution of MIQP-DOMP.

Proof Indeed, let us assume that (9) holds together with the remaining inequalities in the formulation MIQP-DOMP. We shall prove that (35) also holds. We sum $X_{j\ell} \leq \mathcal{O}_\ell$ for all $j = 1 \ldots, n$ which results in

$$\sum_{j=1}^{n} X_{j\ell} \leq n\mathcal{O}_\ell. \tag{36}$$

We assume free self-service and no capacity constraints for the allocation of the demand points to the facilities. Therefore, for each $\mathcal{O}_{\hat{\ell}} = 1$, among all the feasible solutions with the same objective value, there is always one satisfying that $X_{\hat{\ell}\hat{\ell}} = 1$ as well. That is, the demand point at $\hat{\ell}$ will be served by the facility at the same location. Since there are p open facilities, there exist $\mathcal{O}_{\ell_1} = 1, \ldots, \mathcal{O}_{\ell_p} = 1$. This implies that in any sum of the form $\sum_{j=1}^{n} X_{j\ell} \leq n\mathcal{O}_\ell$ there must be at least $p-1$ variables $X_{\ell_s\ell} = 0$, if $\ell_s \neq \ell$, since by (7) there are, at least, $p-1$ open facilities different from ℓ (even if facility at ℓ happens to be open).

The above argument implies that, whenever $\mathcal{O}_\ell = 1$ in (36), we can subtract $p-1$ units from the right-hand-side of the inequality, (obviously, this also holds if $\mathcal{O}_\ell = 0$) resulting in (35).

Conversely, if $\mathcal{O}$ are binary the fact that (35) implies (9) is straightforward. □

These constraints could have been used to modified the formulation MIQP-DOMP replacing the constraints (9) by (35). However, we have not followed this approach because it would have required to consider the variables $\mathcal{O}_\ell$ as binary whereas with our current approach this can be avoided.

Our second observation shows that the objective function (11) of CP-DOMP admits an alternative representation. Indeed, by (23)

$$\begin{aligned}
&\sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} P_{jk} \sum_{\ell=1}^{n} c_{j\ell} X_{j\ell} + 1/2 \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{j'=1}^{n}\sum_{k'=1}^{n} d_{jkj'k'} P_{jk} P_{j'k'} \\
&\quad + 1/2 \sum_{j=1}^{n}\sum_{\ell=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} h_{j\ell pq} X_{j\ell} X_{pq} = 1/2\langle G, \Phi\rangle \\
&\quad = \langle F, V\rangle + 1/2\langle D, Q\rangle + 1/2\langle H, U\rangle.
\end{aligned}$$

Next, using (24) we obtain $\langle F, V\rangle = \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_{j\ell} P_{jk} X_{j\ell} = \sum_{k=1}^{n} \lambda_k W_k = \langle g, \phi\rangle$. Then, for any $\mu \in [0, 1]$, combining both expressions and using that $\Phi\alpha = \phi$ and therefore $W_k = \sum_{\ell=1}^{n} \rho_{1\ell k}$, for all $k = 1, \ldots, n$, we get:

$$\begin{aligned}
\langle F, V\rangle + 1/2\langle D, Q\rangle + 1/2\langle H, U\rangle =& \mu\langle F, V\rangle + (1-\mu)\langle g, \phi\rangle + 1/2\langle D, Q\rangle + 1/2\langle H, U\rangle \\
=& 1/2\langle D, Q\rangle + 1/2\langle H, U\rangle + \mu\langle F, V\rangle + (1-\mu)\langle \lambda, W\rangle \\
=& 1/2\langle D, Q\rangle + 1/2\langle H, U\rangle + \mu\langle F, V\rangle \\
&+ (1-\mu)\sum_{k=1}^{n} \lambda_k \sum_{\ell=1}^{n} \rho_{1\ell k}.
\end{aligned} \tag{37}$$

In other words, the objective function of CP-DOMP can be equivalently written for any $\mu \in [0, 1]$ as:

$$\mu\langle F, V\rangle + (1-\mu)\sum_{k=1}^{n} \lambda_k \sum_{\ell=1}^{n} \rho_{1\ell k} + 1/2\langle D, Q\rangle + 1/2\langle H, U\rangle. \tag{38}$$

We observe from (38) that the objective function of CP-DOMP is also valid for $\mu = 0$ and $\mu = 1$. This makes it possible to avoid V or ρ variables in the objective function.

Furthermore, we would like to remark that 7 out of the 21 blocks of the matrix variables Φ, namely $\mathcal{O}W \in \mathbb{R}^{n\times n}$, $\mathcal{O}\xi \in \mathbb{R}^{n\times n}$, $\mathcal{O}\eta \in \mathbb{R}^{n\times n^2}$, $W\zeta \in \mathbb{R}^{n\times n^2}$, $\xi\eta \in \mathbb{R}^{n\times n^2}$, $\xi\zeta \in \mathbb{R}^{n\times n^2}$ and $\eta\zeta \in \mathbb{R}^{n^2\times n^2}$, never appear explicitly in any constraint of CP-DOMP, except as $\Phi \in \mathcal{C}^*$. It is not clear whether these final remarks may help in solving the problem or not.

5 Concluding remarks

The results in this paper state, for the first time, the equivalence of a difficult NP-hard discrete location problem, namely DOMP, with a continuous, convex optimization problem. This approach can be used to start new avenues of research by applying tools from continuous optimization to approximate or numerically solve some other hard discrete location problems. Some examples are single allocation hub location problems and ordered median hub location problems with and without capacities, see e.g. Fernández et al. [12]; Puerto et al. [23,24], to mention a few. The aim of this paper is not computational but it seems natural to consider some relaxations of formulation CP-DOMP to analyze the accuracy of their related bounds. This is beyond the scope of this contribution but will be the subject of a follow up paper. (The reader is referred to Bomze et al. [5] for further details.)

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Appendix: An explicit formulation of CP-DOMP

First of all, one can check that

$$\begin{aligned}\Phi[a_{(4)}]_1 = \Big[& \overbrace{\sum_{\ell=1}^{n} q_{111\ell}, \ldots, \sum_{\ell=1}^{n} q_{nn1\ell}}^{(PP)}, \overbrace{\sum_{\ell=1}^{n} v_{111\ell}, \ldots, \sum_{\ell=1}^{n} v_{nn1\ell}}^{(PX)}, \overbrace{\sum_{\ell=1}^{n} (P\mathcal{O})_{1\ell 1}, \ldots, \sum_{\ell=1}^{n} (P\mathcal{O})_{1\ell n}}^{(P\mathcal{O})}, \\ & \overbrace{\sum_{\ell=1}^{n} (PW)_{1\ell 1}, \ldots, \sum_{\ell=1}^{n} (PW)_{1\ell n}}^{(PW)}, \overbrace{\sum_{\ell=1}^{n} (P\xi)_{1\ell 1}, \ldots, \sum_{\ell=1}^{n} (P\xi)_{1\ell n}}^{(P\xi)}, \overbrace{\sum_{\ell=1}^{n} (P\eta)_{1\ell 11}, \ldots, \sum_{\ell=1}^{n} (P\eta)_{1\ell nn}}^{(P\eta)}, \\ & \overbrace{\sum_{\ell=1}^{n} (P\zeta)_{1\ell 11}, \ldots, \sum_{\ell=1}^{n} (P\zeta)_{1\ell nn}}^{(P\zeta)} \Big]^T.\end{aligned}$$

Next, we present the explicit formulation of CP-DOMP, which is obtained from the original formulation replacing the original variables ϕ by its linear expression in terms of Φ, namely $\phi = \Phi[a_{(4)}]_1$ (see theorems 3.1 and 3.2).

Indeed, this reformulation requires to include: 1) the linear constraints that come from MIQP1-DOMP rewritten using that $\Phi[a_{(4)}]_1 = \phi$, 2) the squares of those constraints in the matrix variable Φ, 3) the quadratic constraints written in the matrix variables Φ and 4) $\Phi \in \mathcal{C}^*$, the cone of completely positive matrices of the appropriate dimension.

In the following we check that the four conditions mentioned above give us the constraints that appear in the explicit representation of CP-DOMP included below. Indeed,

1. Constraints (39) are $[a_{(4)}]_i^T \Phi [a_{(4)}]_1 = 1$ for all $i = 1, \ldots, n$. Analogously, constraints (40) are $[a_{(5)}]_i^T \Phi [a_{(4)}]_1 = 1$ for all $i = 1, \ldots, n$; constraint (41) is $[a_{(7)}]^T \Phi [a_{(4)}]_1 = p$; constraints (42) are $[a_{(8)}]_i^T \Phi [a_{(4)}]_1 = 1$ for all $i = 1, \ldots, n$; constraints (44) are $[a_{(17)}]_k^T \Phi [a_{(4)}]_1 = 1$ for all $k = 1, \ldots, n-1$; constraints (43) are $[a_{(12)}]_{j\ell}^T \Phi [a_{(4)}]_1 = 1$ for all $j, \ell = 1, \ldots, n$; and constraints (45) are $[a_{(13)}]_{jk}^T \Phi [a_{(4)}]_1 = 1$ for all $j, k = 1, \ldots, n$. This proves that the block $A\phi = A\Phi[a_{(4)}]_1 = b$ appears in CP–DOMP-Explicit.
2. Constraints (46)–(53) are obtained squaring (4),(5), (7),(8), (17), (12), (18) and (13), respectively. This proves that $diag(A^T \Phi A) = b \circ b$ also appears in CP-DOMP-Explicit.
3. Constraints (54), (55) and (56) are the quadratic constraints (14), (15) and (16) replacing the quadratic terms by the corresponding matrix variables in Φ (recall that Φ was introduced in (22)). Hence, CP-DOMP-Explicit includes the quadratic constraints in MIQP1-DOMP written in terms of the matrix variables Φ.
4. $\Phi \in \mathcal{C}^*_{(4n^2+3n)\times(4n^2+3n)}$, the appropriate dimension of the space of variables.

We note in passing that we do not need to add the constraint $[a_{(4)}]_1^T \Phi [a_{(4)}]_1 = 1$ since it is already included. Indeed, it is the first constraint in the block (39).

Based on the above discussion, the explicit reformulation of CP-DOMP as a completely positive convex problem in the essential matrix variables Φ is:

$$\min \quad \langle F, V\rangle + 1/2\langle D, Q\rangle + 1/2\langle H, U\rangle \tag{CP-DOMP-Explicit}$$

$$\text{s.t.} \sum_{j=1}^{n}\sum_{\ell=1}^{n} q_{ij1\ell} = 1, \quad \forall i = 1, \ldots, n, \tag{39}$$

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} q_{ki1\ell} = 1, \quad \forall i = 1, \ldots, n, \tag{40}$$

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} (P\mathcal{O})_{1k\ell} = p, \tag{41}$$

$$\sum_{j=1}^{n}\sum_{\ell=1}^{n} v_{ij1\ell} = 1, \quad \forall i = 1, \ldots, n, \tag{42}$$

$$\sum_{k=1}^{n} v_{1kj\ell} - \sum_{k=1}^{n} (P\mathcal{O})_{1k\ell} + \sum_{k=1}^{n} (P\zeta)_{1kj\ell} = 0, \quad \forall j, \ell = 1, \ldots, n, \tag{43}$$

$$\sum_{\ell=1}^{n} \rho_{1\ell k} - \sum_{\ell=1}^{n} \rho_{1\ell k+1} + \sum_{\ell=1}^{n} (P\xi)_{1\ell k} = 0, \quad \forall k = 1, \ldots, n-1, \tag{44}$$

$$\sum_{\ell=1}^{n} \rho_{1\ell k} - \sum_{\ell=1}^{n} \left(\sum_{r=1}^{n} v_{1rj\ell}\right) c_{j\ell} + \sum_{\ell=1}^{n} c_{j\ell}\left(1 - \sum_{r=1}^{n} q_{jk1r}\right) + \sum_{r=1}^{n} v_{1rjk} = 0, \quad \forall j, k = 1, \ldots, n, \tag{45}$$

$$\sum_{i=1}^{n} q_{ikik} = 1, \quad \forall k = 1, \ldots, n, \tag{46}$$

$$\sum_{i=k}^{n} q_{ikik} = 1, \quad \forall i = 1, \ldots, n, \tag{47}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} = p^2, \tag{48}$$

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} u_{jkj\ell} = 1, \quad \forall j = 1, \ldots, n, \tag{49}$$

$$\omega_{kk} - 2\omega_{k,k+1} + 2\delta_{kk} + \omega_{k+1,k+1} - 2\delta_{k,k+1} + \psi_{kk} = 0, \quad \forall k = 1, \ldots, n-1, \tag{50}$$

$$u_{j\ell j\ell} - 2\chi_{j\ell\ell} + 2\tau_{j\ell j\ell} - 2\beta_{\ell j\ell} + \sigma_{\ell\ell} + z_{j\ell j\ell} = 0, \quad \forall j, \ell = 1, \ldots, n, \tag{51}$$

$$\sum_{r=1}^{n}\sum_{s=1}^{n} c_{rs}u_{rsrs} + \sum_{\substack{i,j,r,s=1\\(i,j)\neq(r,s)}}^{n} c_{ij}c_{rs}u_{ijrs} - 2\sum_{i,j=1} nc_{ij}\left(\sum_{\ell=1}^{n}\gamma_{ij\ell}\right) + \sum_{r,s=1}^{n}\omega_{rs} = 0, \tag{52}$$

$$\begin{aligned}&\left(\sum_{\ell=1}^{n} c_{j\ell}\right)^2 q_{jkjk} + \sum_{\ell=1}^{n} c_{j\ell}^2 u_{j\ell j\ell} - 2\left(\sum_{\ell=1}^{n} c_{j\ell}\right)\sum_{\ell=1}^{n} c_{j\ell}v_{jkj\ell}\\ &\quad + 2\sum_{\ell=1}^{n} c_{j\ell}\rho_{jkk} + 2\left(\sum_{\ell=1}^{n} c_{j\ell}\right)v_{jkjk} + 2\sum_{r<s}^{n} c_{jr}c_{js}u_{jrjs} - 2\sum_{\ell=1}^{n} c_{j\ell}\gamma_{j\ell k}\\ &\quad - 2\sum_{\ell=1}^{n} c_{j\ell}\kappa_{j\ell jk} + 2\epsilon_{kjk} + \omega_{kk} + \pi_{jkjk} = \left(\sum_{\ell=1}^{n} c_{j\ell}\right)^2, \quad \forall j, k = 1, \ldots, n,\end{aligned} \tag{53}$$

$$\sum_{\ell=1}^{n}\rho_{1\ell k} - \sum_{j=1}^{n}\sum_{\ell=1}^{n} c_{j\ell}v_{jkj\ell} = 0, \quad \forall k = 1, \ldots, n, \tag{54}$$

$$\sum_{j,k=1}^{n}\left(\sum_{\ell=1}^{n} q_{1\ell jk} - q_{jkjk}\right) = 0, \tag{55}$$

$$\sum_{j,\ell=1}^{n}\left(\sum_{r=1}^{n} v_{j\ell 1r} - u_{j\ell j\ell}\right) = 0, \tag{56}$$

$$\Phi \in \mathcal{C}^*_{(4n^2+3n)\times(4n^2+3n)}. \tag{57}$$

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